Analysis Preliminary Exam

May 2019

Do all 10 problems. Show your work.

P 1.

1) State the one dimensional Intermediate Value Theorem.

2) State the corresponding result in higher dimensions, giving carefully the definitions of the concepts involved. Why does 2) imply 1)?

P 2.

1) Define the total derivative (differential) of a function $f : D \subseteq \mathbb{R}^m \to \mathbb{R}^n$ at a point x_0 of the open set D.

2) Explain without proof how to obtain the partial derivatives from the total derivative and the condition for the reciprocal. Are they ever equivalent?

P 3.

Define the boundary ∂A of $A \subseteq \mathbb{R}^n$ to be the set of points y such that for any $r > 0$, the intersection of any ball $B(y, r)$ with both A and A^c is not empty. Show that ∂A is a closed set.

P 4.

Let $f : \mathbb{R} \to \mathbb{R}$ satisfy, for a positive C

$$\forall x, y \qquad |f(x) - f(y)| \leq C|x - y|^p\,, \qquad p > 0\,.$$

1) Show that f is uniformly continuous.

2) Show that if $p = 1$ and f differentiable, then its derivative is bounded.

3) Show that if $p > 1$ the function has derivative equal to zero.

P 5.

Let

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{n} \left(\frac{x}{x+4} \right)^n .$$

1) Determine $x \in \mathbb{R}$ such that $f(x)$ converges.

2) When $x > 0$, calculate $f(x)$, justifying your steps.

P 6.

Let $f_n : [0,1] \to \mathbb{R}$ be continuous, equal to zero on $\{0\} \cup [\frac{1}{n}, 1]$, $f_n(\frac{1}{2n}) = 2n$ and linear on $[0, \frac{1}{2n}]$ and $[\frac{1}{2n}, \frac{1}{n}]$.

a) Show that $f_n(x)$ converges pointwise to a continuous function.

b) Show that the integrals of f_n do not converge to the integral of the limit.

c) Does f_n converge uniformly?

P 7.

Assume $f : \mathbb{R} \to \mathbb{R}$ has a uniformly continuous derivative. Then

$$U(x) = \int_0^1 f(tx)dt$$

is well defined on $\mathbb{R}$. Calculate rigorously $U'(x)$.

P 8.

Let $F : \mathbb{R}^d \to \mathbb{R}$, $\gamma : \mathbb{R} \to \mathbb{R}^d$ have continuous derivatives and $F(\gamma(t)) = c$, $c \in \mathbb{R}$ for all $t \in \mathbb{R}$.

1) Show that the tangent vector to γ at $x_0 = \gamma(t_0)$ belongs to the tangent plane to $S = \{x | F(x) = c\}$.

2) Verify that if $F(x) = g(||x||)$, g smooth and increasing, then $\gamma(t) \cdot \gamma'(t) = 0$ for all t.

P 9.

On $K = \{(x,y) | x^2 - xy + y^2 \leq 50\}$, define

$$f(x,y) = x^3 + 6x^2 + 3y^2 - 12xy + 9x\,.$$

1) Justifiy that f achieves its absolute extrema on K.

2) Find the local extrema/ saddle points in $int(K)$.

3) Sketch how to find the extrema on ∂K. Do not calculate.

P 10.

Let

$$(x,y,z) \in \mathbb{R}^3 \to F(x,y,z) = (yz, xz, xy)\ .$$

1) Determine the points where F has a local continuous inverse.

2) Calculate $(F^{-1})'(1,1,1)$ with the formula in the inverse function theorem.

3) Calculate the inverse function directly, i.e. solve $(u,v,w) = F(x,y,z)$ for (x,y,z).